

$$[\tilde{x}_i, \tilde{x}_j] = 0, \quad [\tilde{p}_i, \tilde{p}_j] = 0, \quad [\tilde{x}_i, \tilde{p}_j] = i\hbar \delta_{ij}$$

other useful identities.

$$\bullet [A, A] = 0, \quad [A, B] = -[B, A], \quad [A, c] = 0 \quad \leftarrow \text{c-number}$$

$$\bullet [A+B, C] = [A, C] + [B, C]$$

$$\bullet [A, BC] = [A, B]C + B[A, C]$$

$$\bullet [A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$$

(Jacobi identity)

## 1.7 Wave functions in position and momentum space.

### (1) Position-Space wave function

$$\rightarrow \text{Base kets} = \text{"position" kets} : \tilde{x}|x\rangle = x|x\rangle$$

$$\text{orthogonality} : \langle x|x'\rangle = \delta(x-x')$$

|| completeness rel.

$$\rightarrow \text{Wave function}$$

$$\int dx |x\rangle\langle x| = 1$$

$$\text{a physical state } |\alpha\rangle = \int dx |x\rangle\langle x|\alpha\rangle$$

$$= \int dx \psi_\alpha(x) |x\rangle$$

- Wave function in position space  $\nearrow \approx$  expansion coefficient of  $x$ -ket "localized" at  $x$ .

$$\psi_\alpha(x) = \langle x|\alpha\rangle$$

- Inner product

$$\langle \beta|\alpha\rangle = \int dx \langle \beta|x\rangle\langle x|\alpha\rangle$$

$$= \int dx \psi_\beta^*(x) \psi_\alpha(x)$$

|| probability for the particle to be found in  $[x, x+dx]$

$$= |\psi_\alpha|^2 dx$$

• Change of base kets

36

$$|\alpha\rangle = \sum_a |a\rangle \langle a|\alpha\rangle \quad \dots \text{What about the Wave functions?}$$

$$\hookrightarrow \langle x|\alpha\rangle = \sum_a \langle x|a\rangle \langle a|\alpha\rangle$$

$$\rightarrow \psi_\alpha(x) = \sum_a c_a u_a(x)$$

For example, <sup>choosing</sup>  $|a\rangle$  to be the eigenket of Hamiltonian,

then  $u_a(x)$  is the eigenfunction ( $\langle x|a\rangle$ ).

and  $\psi_\alpha(x)$  is a general form of wave function.

(a particle in a box:  $u_1(x) = e^{ikx}$ ,  $u_2 = e^{-ikx}$ )

$$\psi(x) = c_1 e^{ikx} + c_2 e^{-ikx}$$

• matrix element  $\langle \beta|A|\alpha\rangle$  of an operator  $A$ .

$$\langle \beta|A|\alpha\rangle = \int dx' \int dx'' \langle \beta|x'\rangle \langle x'|A|x''\rangle \langle x''|\alpha\rangle$$

$$= \int dx' \int dx'' \psi_\beta^*(x') \langle x'|A|x''\rangle \psi_\alpha(x'')$$

ex.  $A = \tilde{x}^2$

$$\langle x'| \tilde{x}^2 |x''\rangle = x'^2 \langle x'|x''\rangle = x'^2 \delta(x'-x'')$$

$$\langle \beta|x^2|\alpha\rangle = \int dx' \int dx'' \psi_\beta^*(x') \underbrace{x'^2}_{\text{do it}} \underbrace{\delta(x'-x'')} \psi_\alpha(x'')$$

$$= \int dx' \psi_\beta^*(x') x'^2 \psi_\alpha(x')$$

In general,  $\langle \beta|f(\tilde{x})|\alpha\rangle = \int dx \psi_\beta^*(x) f(x) \psi_\alpha(x)$

Let's play with the infinitesimal translation operation.

$$\begin{aligned}
 \left(1 - \frac{i\tilde{p}\delta x}{\hbar}\right)|\alpha\rangle &= \int dx' J(\delta x) |x'\rangle \langle x'|\alpha\rangle \\
 &= \int dx' |x' + \delta x\rangle \langle x'|\alpha\rangle \quad \xrightarrow{x' \rightarrow x' - \delta x} \\
 &= \int dx' |x'\rangle \langle x' - \delta x|\alpha\rangle \\
 &\quad \downarrow \text{Taylor series expansion} \\
 &= \int dx' |x'\rangle \left[ \langle x'|\alpha\rangle - \delta x \frac{\partial}{\partial x'} \langle x'|\alpha\rangle + \cancel{\frac{(\delta x)^2}{2}} \dots \right] \\
 \text{* } f(x) &\equiv \langle x|\alpha\rangle \\
 f(x \pm \delta x) &= f(x) + \frac{\partial}{\partial x} f(x) \bigg|_{x'} (x - x') + \dots \\
 f(x' + \delta x) &= f(x') - \delta x \frac{\partial}{\partial x'} f(x') + \dots
 \end{aligned}$$

$$\Rightarrow |\alpha\rangle - \frac{i\delta x}{\hbar} \tilde{p}|\alpha\rangle = \int dx' |x'\rangle \langle x'|\alpha\rangle - \delta x \int dx' |x'\rangle \frac{\partial}{\partial x'} \langle x'|\alpha\rangle$$

$$\Rightarrow \tilde{p}|\alpha\rangle = \int dx' \int dx'' |x'\rangle \langle x'|\tilde{p}|x''\rangle \langle x''|\alpha\rangle \quad \uparrow \text{compare!}$$

$$\therefore \langle x'|\tilde{p}|x''\rangle = -i\hbar \frac{\partial}{\partial x'} \delta(x' - x'')$$

• For general kets  $|\alpha\rangle$  and  $|\beta\rangle$ ,

$$\begin{aligned}
 \langle \beta|\tilde{p}|\alpha\rangle &= \int dx' \int dx'' \langle \beta|x'\rangle \langle x'|\tilde{p}|x''\rangle \langle x''|\alpha\rangle \\
 &= \int dx' \int dx'' \psi_{\beta}^*(x') \cdot \left( -i\hbar \frac{\partial}{\partial x'} \delta(x' - x'') \right) \cdot \psi_{\alpha}(x'') \\
 &\quad \uparrow \text{integrate first.} \\
 &= \int dx' \psi_{\beta}^*(x') \left( -i\hbar \frac{\partial}{\partial x'} \right) \psi_{\alpha}(x')
 \end{aligned}$$

• For  $\tilde{p}^n$ ,  $\langle x'|\tilde{p}^n|\alpha\rangle = \left( -i\hbar \frac{\partial}{\partial x'} \right)^n \langle x'|\alpha\rangle$

$$\langle \beta|\tilde{p}^n|\alpha\rangle = \int dx' \psi_{\beta}^*(x') \left[ -i\hbar \frac{\partial}{\partial x'} \right]^n \psi_{\alpha}(x')$$

### (3) Momentum - Space wave function.

38

Similarly for the base eigenkets in the  $\tilde{p}$ -basis,

$$\tilde{p}|p\rangle = p|p\rangle, \quad \langle p'|p\rangle = \delta(p'-p)$$

completeness relation  $1 = \int dp |p\rangle \langle p|$  "

• momentum-space wave function.

$$\langle p|\alpha\rangle = \phi_\alpha(p) \quad \text{as} \quad |\alpha\rangle = \int dp |p\rangle \langle p|\alpha\rangle \quad \nearrow \phi_\alpha(p)$$

• transformation :  $\alpha \longleftrightarrow p$  ( $\langle x|p\rangle$ )

$$\text{we have } \langle x|\tilde{p}|\alpha\rangle = -i\hbar \frac{\partial}{\partial x} \langle x|\alpha\rangle.$$

Putting  $|p\rangle \rightarrow |\alpha\rangle$ , then  $\langle x^0|\tilde{p}|p\rangle = -i\hbar \frac{\partial}{\partial x} \langle x|p\rangle$

$$\therefore p \langle x|p\rangle = -i\hbar \frac{\partial}{\partial x} \langle x|p\rangle \quad \dots \text{first-order diff. eq.}$$

$$\Rightarrow \langle x|p\rangle = N \exp\left[\frac{i p x}{\hbar}\right] \quad \parallel \quad N = \text{normalization constant.}$$

To get  $N$ , use

$$\delta(x'-x'') = \langle x'|x''\rangle = \int dp \langle x'|p\rangle \langle p|x''\rangle$$

$$= |N|^2 \int dp \exp\left[\frac{i p (x'-x'')}{\hbar}\right]$$

$$= 2\pi\hbar |N|^2 \delta(x'-x'')$$

$$N = \frac{1}{\sqrt{2\pi\hbar}}$$

$$\therefore \langle x^0|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i p x}{\hbar}}$$

|| "definition of  $\delta(x)$

$$\delta(x) = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dk e^{ikx}$$

⇒ Transformations.

"It's F.T.  
and inverse F.T."

•  $\langle x | \alpha \rangle = \int dp \langle x | p \rangle \langle p | \alpha \rangle = \mathcal{D}$

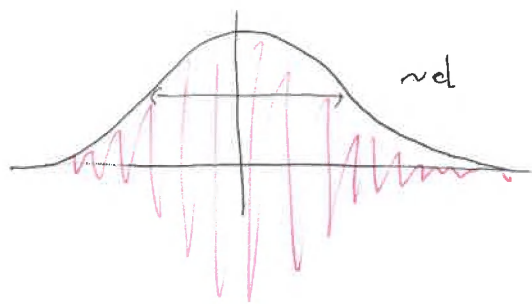
$$\psi_\alpha(x) = \frac{1}{\sqrt{2\pi\hbar}} \int dp e^{i \frac{px}{\hbar}} \phi_\alpha(p)$$

•  $\langle p | \alpha \rangle = \int dx \langle p | x \rangle \langle x | \alpha \rangle \Rightarrow$

$$\phi_\alpha(p) = \frac{1}{\sqrt{2\pi\hbar}} \int dx e^{-i \frac{px}{\hbar}} \psi_\alpha(x)$$

### (4) Gaussian Wave Packets.

• In Position space,  $\langle x | \alpha \rangle = \left[ \frac{1}{\pi^{1/4} \sqrt{d}} \right] \exp \left[ i k x - \frac{x^2}{2d^2} \right]$



: "minimum" uncertainty wave packet.

$$\langle \Delta \tilde{x}^2 \rangle \langle \Delta \tilde{p}^2 \rangle = \frac{\hbar^2}{4}$$

• Expectation values :

$$\langle \tilde{x} \rangle = \int_{-\infty}^{\infty} dx \langle \alpha | x \rangle x \langle x | \alpha \rangle = 0.$$

$$\langle \tilde{x}^2 \rangle = \frac{d^2}{2}, \quad \rightarrow \langle (\Delta \tilde{x})^2 \rangle = \langle \tilde{x}^2 \rangle - \langle \tilde{x} \rangle^2 = \frac{d^2}{2}$$

$$\langle \tilde{p} \rangle = \hbar k, \quad \langle \tilde{p}^2 \rangle = \frac{\hbar^2}{2d^2} + \hbar^2 k^2.$$

$$\rightarrow \langle (\Delta \tilde{p})^2 \rangle = \langle \tilde{p}^2 \rangle - \langle \tilde{p} \rangle^2 = \frac{\hbar^2}{2d^2}$$

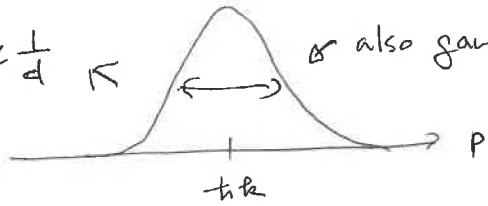
• momentum-space wave function :

$$\langle p | \alpha \rangle = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dx e^{-i \frac{px}{\hbar}} \langle x | \alpha \rangle.$$

~~$$\neq \sqrt{\frac{A}{\hbar}} \exp \left[ \frac{p^2}{\hbar} \right]$$~~

$$\langle p | \alpha \rangle = \frac{1}{\sqrt{2\pi\hbar}} \left( \frac{1}{\pi^{1/4} d} \right) \int_{-\infty}^{\infty} dx \exp \left( -\frac{i}{\hbar} (p - \hbar k) x - \frac{x^2}{2d^2} \right) \quad 40.$$

$$= \sqrt{\frac{d}{\hbar\sqrt{\pi}}} \exp \left[ -\frac{d^2}{2\hbar^2} (p - \hbar k)^2 \right] \quad \left\| \begin{aligned} & \int_{-\infty}^{\infty} dx e^{-ax^2 - bx} \\ &= \int_{-\infty}^{\infty} dx e^{-a(x + \frac{b}{2a})^2 + \frac{b^2}{4a}} \\ &= \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}} \end{aligned} \right.$$

$\propto \frac{1}{d}$   also gaussian

As it gets broaden in x-space, it gets sharpen in p-space!

ex.  $d \rightarrow \infty$  : infinite uncertainty in x-space.

$\rightarrow \delta$ -function in p-space.

(i.e. momentum is sharply defined!)  
 $\rightarrow$  eigenstate!

(5) Generalization to 3D : trivial!

$$\checkmark \quad \vec{x} | \vec{x} \rangle = \vec{x} | \vec{x} \rangle, \quad \vec{p} | \vec{p} \rangle = \vec{p} | \vec{p} \rangle$$

indication of 3D "

$$\checkmark \quad \langle \vec{p}' | \vec{p}'' \rangle = \delta^{(3)}(\vec{p}' - \vec{p}'') \quad , \quad \langle \vec{x}' | \vec{x}'' \rangle = \delta^{(3)}(\vec{x}' - \vec{x}'')$$

$$\checkmark \quad \delta^3(\vec{x}' - \vec{x}'') \equiv \delta(x' - x'') \delta(y' - y'') \delta(z' - z'')$$

$$\bullet \text{ Completeness: } \int d^3x | \vec{x} \rangle \langle \vec{x} | = 1, \quad \int d^3p | \vec{p} \rangle \langle \vec{p} | = 1$$

$$\bullet \quad \langle \vec{p} | \vec{p} \rangle = \int d^3x \psi_{\vec{p}}^*(\vec{x}) (-i\hbar \nabla) \psi_{\vec{p}}(\vec{x})$$

$$\langle \vec{x} | \vec{p} \rangle = \left[ \frac{1}{(2\pi\hbar)^{3/2}} \right] \exp \left( \frac{i \vec{p} \cdot \vec{x}}{\hbar} \right)$$

$$\psi_{\vec{p}}(\vec{x}) = \frac{1}{(2\pi\hbar)^{3/2}} \int d^3p e^{\frac{i \vec{p} \cdot \vec{x}}{\hbar}} \phi_{\vec{p}}(\vec{p}), \quad \phi_{\vec{p}}(\vec{p}) = \frac{1}{(2\pi\hbar)^{3/2}} \int d^3x e^{-\frac{i \vec{p} \cdot \vec{x}}{\hbar}} \psi_{\vec{p}}(\vec{x})$$